GENERAL PRINCIPLES FOR ASYMPTOTIC CALCULATION OF THE INTERACTION BETWEEN CHARGES AND SPATIALLY MODULATED MAGNETIC FIELDS

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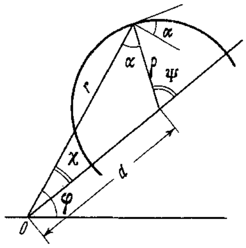
Here we consider the general case of the Krylov-Bogolyubov method applied to the motion of charges in a relatively strong homogeneous magnetic field together with a certain small perturbation, whose form is not specified but which may be dependent on all three spatial coordinates. We use a cylindrical coordinate system (r, φ, z), with the z-axis parallel to the strong field. It is shown that the problem may be reduced to solution of a quasi-harmonic equation whose coefficients are dependent on two slowly varying parameters, whose variations are described by two independent first-order equations. The three equations form a system to which we may apply the usual methods of the asymptotic theory of nonlinear oscillations, in particular the method of solution described in [1] (§13 of chapter III).



We assume that the components of the magnetic field may be expressed as

$$H_z = H_0\,[1 + \varepsilon h_z\,(r, \varphi, z)],$$

$$H_r = \varepsilon H_0 h_r\,(r, \varphi, z), \quad H_\varphi = \varepsilon H_0 h_\varphi\,(r, \varphi, z).$$

We substitute these into the equations for the motion of a charge e having a mass m and put $\omega_0 = eH_0/mc$ to get [2]

$$\ddot{r} - r\dot{\varphi}^2 = -\,\omega_0\,\{r\dot{\varphi} + \varepsilon\,(r\dot{\varphi}h_z - \dot{z}h_\varphi)\},$$

$$\frac{d}{dt}\,(r^2\dot{\varphi}) = \omega_0\,\{r\dot{r} + \varepsilon r\,(\dot{z}h_r - \dot{r}h_z)\}.$$

We put

$$\dot{\varphi} = {}^1\!/_2\,\omega_0 + \theta\,/\,r^2, \qquad (1)$$

in which θ is a new unknown function; then (1) becomes

$$\ddot{r} + {}^1\!/_4\,\omega_0^2 r - \theta^2\,/\,r^3 = \varepsilon\omega_0\,\{vh_\varphi - r\,({}^1\!/_2\,\omega_0 + \theta\,/\,r^2)\,h_z\}$$

$$\dot{\theta} = \varepsilon\omega_0 r\,(vh_r - \dot{r}h_z),$$

$$\dot{v} = \varepsilon\omega_0\,\{r\,({}^1\!/_2\,\omega_0 + \theta\,/\,r^2)\,h_r - \dot{r}h_\varphi\}\;(v \equiv \dot{z}). \qquad (2)$$

This shows that θ and v are slowly varying functions of time. It is readily shown that θ is constant at ${}^1\!/_2\,\omega_0(\rho^2 - d^2)$ for a constant homogeneous field, in which ρ is the Larmov radius and d is the distance from the center of that orbit to the z axis. Figure 1 shows that the change in φ over a short time is

$$\Delta\varphi = \frac{\rho}{r}\,\Delta\psi\cos\alpha = \frac{\rho\,(\rho + \alpha\cos\psi)}{r^2}\,\Delta\psi.$$

We then make the substitution $r^2 = \rho^2 + d^2 + 2\rho d\cos\psi$ and some elementary transformations to get

$$\dot{\varphi} = \left(\frac{1}{2} + \frac{\rho^2 - d^2}{2r^2}\right)\dot{\psi} = \frac{1}{2}\left(1 + \frac{\rho^2 - d^2}{r^2}\right)(\omega_0 + \dot{P}) \qquad (3)$$

in which $\dot{P}$ is a small quantity, since it must tend to zero along with ε. Comparison of (3) with (1) gives

$$\theta = {}^1\!/_2\,\omega_0\,(\rho^2 - d^2) + \dot{P}\,\rho\,(\rho + d\cos\psi). \qquad (4)$$

We isolate from φ the rapidly varying part $\chi = \arcsin(\rho\sin\psi/r)$ and denote $\varphi - \chi$ by η. Then (3) allows us to show that

$$\dot{\eta} = r^{-2}\,(\rho\dot{d} - \dot{\rho}d)\sin\psi,$$

so to an accuracy of the first order we have

$$\eta = \sigma + \frac{\dot{\rho}d - \rho\dot{d}}{\omega_0\rho d}\ln r, \qquad (5)$$

in which σ is an arbitrary constant, which may, however, be taken as less than 2π. This means that $\varphi = \sigma + \chi(\psi)$ within the framework of the first approximation, since all terms dependent on φ in the equations of motion are multiplied by ε, while the second term on the right in (5) may be disregarded, provided that r does not become zero; to avoid the latter, we must rule out the case $|\rho - d| \lesssim \varepsilon$, since $r \approx \varepsilon$ for $|\rho - d| \approx \varepsilon$, while the second term on the right in (5) still remains of order $\varepsilon\ln\varepsilon$.

Then φ on the right in (2) may be replaced everywhere as follows:

$$\varphi = \sigma + \arcsin\,[\rho\sin\psi\,/\,r].$$

It is often more important to know how the parameters of the motion vary with z (not with t), so we convert in (2) from differentiation with respect to t to differentiation with respect to z, denoting the latter by a prime, i.e., $r' = \partial r/\partial z$, etc. Then

$$\ddot{r} = r''v^2 + r'\dot{v}, \quad |\,r'\dot{v}\,| \ll |\,r''v^2\,| \quad (\dot{v} \sim \varepsilon).$$

Then the $r'\dot{v}$ term in the first equation of (2) should be transferred to the right, while $\dot{v}$ is replaced by the right-hand part of the third equation in (2). We also put $\Omega(v) = \omega_0/2v$ to get in place of (2)

$$r'' + \Omega^2 r - \left(\frac{\theta}{v}\right)^3\frac{1}{r^3} = \varepsilon 2\Omega\left\{h_\varphi - r\left(\Omega + \frac{\theta}{vr^3}\right)h_z\right\} - \frac{r'v'}{r}$$

$$\theta' = \varepsilon\omega_0 r\,(h_r - r'h_z), \qquad v' = \varepsilon\omega_0\left\{r\left(\Omega + \frac{\theta}{vr^2}\right)h_r - r'h_\varphi\right\}. \qquad (6)$$

We now introduce instead of r a new function τ related to r as follows:

$$r = \sqrt{\rho^2 + d^2 + \tau} \quad \text{or} \quad \tau = 2\rho d\cos\psi, \qquad (7)$$

in which in transferring to differentiation with respect to z we put

$$\psi = \int \Omega\,(v)\,dz + \Phi\,(z),$$

in which $\Phi(z)$ is a slowly varying function of z. We put

$$a = 2\,\rho d, \quad b = \rho^2 + d^2. \qquad (8)$$

We substitute (7) into the first equation in (6) and differentiate with respect to z; all small quantities are then transferred to the right, and the equation is divided by $2r^2 = 2(\tau + b)$, which gives

$$\frac{d}{dz}\,(\tau'' + \Omega^2\tau) = \frac{2}{r^2}\,\frac{d}{dz}\,[r^3\varepsilon F\,(\ldots)] +$$

$$+ \frac{1}{2}\,(\Omega^2)'\,(\tau - b) - b''' - \Omega^2 b' + \frac{2}{r^2}\left(\frac{\theta^2}{v^2}\right)', \qquad (9)$$

in which $\varepsilon F(\ldots)$ is the right part of the first equation of (6). The derivatives of θ and v are replaced by the right-hand parts of (6), while b may be eliminated via (8) and (4), which gives

$$b = b_0 - 2\frac{P^{\cdot}\theta}{\omega_0^2}\left\{1 + \frac{2}{b_0}\left(\frac{\theta}{\omega_0} + \frac{a}{2}\cos\psi\right)\right\},$$

$$b_0 = \left(a^2 + 4\frac{\theta^2}{\omega_0^2}\right)^{1/2}.$$

As $P^{\cdot}$ is small, only cos ψ within the braces needed be differentiated; then near resonance, where $(\Omega^2 - \nu^2)$ is small, we get in the first approximation from (9) that

$$\frac{d}{dz}(r'' + \Omega^2\tau) = \frac{2}{r^2}\frac{d}{dz}[\varepsilon r^3 F(\ldots)] + \frac{1}{2}(\Omega^2)'(\tau - b_0) - b_0' + \frac{2}{r^2}\left(\frac{\theta^2}{v^2}\right). \quad (10)$$

The linearity of the left side allows us to solve the equation by the usual asymptotic methods, i.e., to put $\tau = a\cos\psi + \varepsilon u_1(\ldots)$ and find a and $\Phi - \nu z$ from

$$\frac{da}{dz} = \varepsilon A_1(a, v, \theta, \Phi), \quad \frac{d\Phi}{dz} = \Omega(v) - \nu + \varepsilon B_1(a, v, \theta, \Phi), \quad (11)$$

in which $2\pi/\nu$ is the period of the perturbation along the z axis.

Equations (11) are solved together with the equations describing the slow variation in v and θ ([1], §13, ch. III). The third order of (11) only slightly complicates the determination of $A_1(a, v, \theta, \Phi)$ and $B(a, v, \theta, \Phi)$; there are no other significant changes in the calculation, which is performed without the assumption of paraxial motion or of the smallness of the energy of the transverse motion.

REFERENCES

1. Yu. A. Mitropol'skii, Problems in the Asymptotic Theory of Nonstationary Oscillations [in Russian], Izd. Nauka, 1964.

2. N. S. Zinchenko, Lectures on Electron Optics [in Russian], Izd. Khar'k. un-ta, 1961.

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